# Boundary-Value Problems

**Boundary-Value Problem (BVP)**: The solution of an ordinary differential equation which must satisfy certain conditions specified for two or more values of the independent variables.

A condition or equation is said to be *homogeneous* if, when it is satisfied by a particular function, $y(x)$, it is also satisfied by $cy(x)$, where $c$ is an arbitrary constant. Here, we are mainly concerned with *homogeneous linear differential equations and associated homogeneous boundary conditions*.

For illustration purpose, let's consider the following homogeneous linear differential equation of 2nd order:

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1}$$

with boundary conditions

$$y(a) = 0 \qquad y(b) = 0 \tag{2}$$

The general solution of equation (1) is of the form

$$y(x) = c_1 u_1(x) + c_2 u_2(x) \tag{3}$$

where $u_1$ and $u_2$ are linearly independent. Substituting equation (3) into equation (2) yields

$$c_1 u_1(a) + c_2 u_2(a) = 0 \tag{4}$$

$$c_1 u_1(b) + c_2 u_2(b) = 0 \tag{5}$$

Let

$$\mathcal{D} = \begin{vmatrix} u_1(a) & u_2(a) \\ u_1(b) & u_2(b) \end{vmatrix} \tag{6}$$

If $\mathcal{D} \neq 0$, then $c_1 = c_2 = 0$. Thus, the solution is trivial! If $\mathcal{D} = 0$, then

$$\frac{u_1(a)}{u_2(a)} = \frac{u_1(b)}{u_2(b)} = -\frac{c_2}{c_1} \tag{7}$$

Thus, the solution can be written as

$$y(x) = c_2\left[\frac{c_1}{c_2}u_1(x) + u_2(x)\right] = c_2\left[-\frac{u_2(a)}{u_1(a)}u_1(x) + u_2(x)\right] \tag{8}$$

$$= C\left[u_2(a)u_1(x) - u_1(a)u_2(x)\right] \tag{9}$$

where $C$ is an arbitrary constant.

In many cases, one or both of the functions $a_1(x)$ and $a_2(x)$ are dependent upon an unspecified parameter $\lambda$, i.e.,

$$\frac{d^2y}{dx^2} + a_1(x, \lambda)\frac{dy}{dx} + a_2(x, \lambda)y = 0 \tag{10}$$

Thus, the solution becomes

$$y(x) = c_1 u_1(x, \lambda) + c_2 u_2(x, \lambda) \tag{11}$$

Therefore, the requirement for equation (10) to have non-trivial solution is

$$\begin{vmatrix} u_1(a, \lambda) & u_2(a, \lambda) \\ u_1(b, \lambda) & u_2(b, \lambda) \end{vmatrix} = 0 \tag{12}$$

Usually, more than one value of $\lambda$ can be found to satisfy equation (12), i.e., $\lambda = \lambda_1, \lambda_2, \cdots$. They are refereed to as the *characteristic values* or *eigenvalues*. The corresponding solutions are called *characteristic functions*.

**[Example]**

$$y'' + \lambda y = 0$$

$$y(0) = 0 \qquad y(L) = 0$$

General solution:

$$y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Apply BCs:

$$x = 0 \qquad y = A = 0$$

$$x = L \qquad y = B\sin\sqrt{\lambda}L = 0$$

Thus,

$$\sqrt{\lambda}L = n\pi \qquad n = 0, 1, 2, \cdots$$

$$\lambda_n = \frac{n^2\pi^2}{L^2} = \text{ characteristic values}$$

$$y = B\sin\frac{n\pi}{L}x$$

$$\varphi_n(x) = \sin\frac{n\pi}{L}x = \text{ characteristic functions}$$

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## Orthogonality of Characteristic Functions

Two function $\varphi_m(x)$ and $\varphi_n(x)$ are said to be *Orthogonal* over an interval $[a, b]$, if

$$\int_a^b \varphi_m(x)\varphi_n(x)dx = 0 \tag{13}$$

They are orthogonal with respect to a weighting function $\tilde{r}(x)$ over an interval $[a, b]$ if

$$\int_a^b \tilde{r}(x)\varphi_m(x)\varphi_n(x)dx = 0 \tag{14}$$

A set of functions, $\{\varphi_k(x)|k = 1, 2, \cdots\}$, is said to be orthogonal in $[a, b]$, if all pairs of distinct functions in the set are orthogonal in $[a, b]$.

Consider the BVP which involves a linear homogeneous 2nd-order differential equation:

$$\frac{d}{dx}\left[\tilde{p}(x)\frac{dy}{dx}\right] + \left[\tilde{q}(x) + \lambda\tilde{r}(x)\right]y = 0 \tag{15}$$

where $\tilde{p}(x)$, $\tilde{q}(x)$ and $\tilde{r}(x)$ are assumed to be real. Define an operator $\mathcal{L}$ as

$$\mathcal{L} \equiv \frac{d}{dx}\left(\tilde{p}\frac{d}{dx}\right) + \tilde{q} \tag{16}$$

Equation (15) can thus be written as

$$\mathcal{L}y + \lambda\tilde{r}(x)y = 0 \tag{17}$$

It should be noted that any 2nd-order ODE of the form

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + \left[a_2(x) + \lambda a_3(x)\right]y = 0 \tag{18}$$

can be transformed to equation (15) by making the following substitution:

$$\tilde{p}(x) = \exp\left[\int \frac{a_1(x)}{a_0(x)}dx\right] \tag{19}$$

$$\tilde{q}(x) = \frac{a_2(x)}{a_0(x)}\tilde{p}(x) \tag{20}$$

$$\tilde{r}(x) = \frac{a_3(x)}{a_0(x)}\tilde{p}(x) \tag{21}$$

Our concern here is to find the required boundary conditions of equation (15), such that the characteristic functions are orthogonal to each other. Let $\lambda_1$ and $\lambda_2$ be two distinct eigenvalues of equation (15) and $\varphi_1(x)$ and $\varphi_2(x)$ be the corresponding eigenfunctions,

$$\frac{d}{dx}\left[\tilde{p}(x)\frac{d\varphi_1}{dx}\right] + \left[\tilde{q}(x) + \lambda_1 \tilde{r}(x)\right]\varphi_1 = 0 \tag{22}$$

$$\frac{d}{dx}\left[\tilde{p}(x)\frac{d\varphi_2}{dx}\right] + \left[\tilde{q}(x) + \lambda_2 \tilde{r}(x)\right]\varphi_2 = 0 \tag{23}$$

$\varphi_2 \times (22) - \varphi_1 \times (23)$

$$\varphi_2 \frac{d}{dx}\left(\tilde{p}\frac{d\varphi_1}{dx}\right) - \varphi_1 \frac{d}{dx}\left(\tilde{p}\frac{d\varphi_2}{dx}\right) + (\lambda_1 - \lambda_2)\tilde{r}(x)\varphi_1\varphi_2 = 0 \tag{24}$$

$$(\lambda_2 - \lambda_1)\int_a^b \tilde{r}(x)\varphi_1(x)\varphi_2(x)dx = \int_a^b \left[\varphi_2 \frac{d}{dx}\left(\tilde{p}\frac{d\varphi_1}{dx}\right) - \varphi_1 \frac{d}{dx}\left(\tilde{p}\frac{d\varphi_2}{dx}\right)\right] dx \tag{25}$$

Integration by parts

$$RHS = \left[\varphi_2\left(\tilde{p}\frac{d\varphi_1}{dx}\right) - \varphi_1\left(\tilde{p}\frac{d\varphi_2}{dx}\right)\right]_a^b - \int_a^b \left[\frac{d\varphi_2}{dx}\left(\tilde{p}\frac{d\varphi_1}{dx}\right) - \frac{d\varphi_1}{dx}\left(\tilde{p}\frac{d\varphi_2}{dx}\right)\right] dx \tag{26}$$

Notice that

$$\int_a^b \left[\frac{d\varphi_2}{dx}\left(\tilde{p}\frac{d\varphi_1}{dx}\right) - \frac{d\varphi_1}{dx}\left(\tilde{p}\frac{d\varphi_2}{dx}\right)\right] dx = 0$$

Since the second term of RHS is zero,

$$(\lambda_2 - \lambda_1) \int_a^b \tilde{r}(x)\varphi_1(x)\varphi_2(x)dx = \left\{ \tilde{p}(x) \left[ \varphi_2(x)\frac{d\varphi_1(x)}{dx} - \varphi_1(x)\frac{d\varphi_2(x)}{dx} \right] \right\}_a^b \tag{27}$$

Therefore, the requirements for $\int_a^b \tilde{r}(x)\varphi_1(x)\varphi_2(x)dx = 0$ are:

- The the RHS of equation (27) vanishes independently at $x = a$ and $x = b$, i.e.,

  $$y(x) = 0 \tag{28}$$

  or

  $$\frac{dy}{dx} = 0 \tag{29}$$

  or

  $$y + \alpha\frac{dy}{dx} = 0 \tag{30}$$

  at $x = a$ or $x = b$. Equations (28) - (30) are referred to as the *Sturm-Liouville Conditions*.

- The RHS of equation (27) will vanish at $x = a$ or $x = b$ when $\tilde{p}(x) = 0$, $y(x)$ is finite and $y'(x)$ is finite (or $\tilde{p}(x)y'(x) \to 0$) at $x = a$ or $x = b$.

- The RHS of equation (27) will be cancelled out when

  $$\tilde{p}(a) = \tilde{p}(b) \tag{31}$$
  $$y(a) = y(b) \tag{32}$$
  $$y'(a) = y'(b) \tag{33}$$

  In other words, $\varphi_1(x)$ and $\varphi_2(x)$ are periodic, of period $(b - a)$.

**[Example]**

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

BCs

$$y(0) = y(L) = 0$$

Notice that they are Sturm-Liouville conditions, i.e., equation (28). Therefore,

$$\tilde{p}(x) = 1 \qquad \tilde{q}(x) = 0 \qquad \tilde{r}(x) = 1$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

$$\varphi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

which have already been obtained before. Next, let's verify orthogonality:

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \frac{L}{2\pi}\left[\frac{1}{m-n}\sin\frac{(m-n)\pi}{L}x - \frac{1}{m+n}\sin\frac{(m+n)\pi}{L}\right]_0^L = 0$$

$$(m \neq n)$$

$$\int_0^L \sin^2\frac{n\pi x}{L}dx = \frac{L}{2} > 0 \qquad (m = n)$$

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If $\tilde{r}(x) > 0$ in $[a, b]$, then $C_n = \int_a^b \tilde{r}(x)\varphi_n^2(x)dx > 0$. Thus, if the multiplication factor is introduced into $\varphi_n(x)$ such that $\bar{\varphi}_n = \varphi_n/\sqrt{C_n}$, then $\bar{\varphi}_n(x)$ is said to be *normalized* w.r.t. $\tilde{r}(x)$. A set of normalized orthogonal functions is said to be *orthonormal*.

**[Example]**

$$C_n = \int_0^L \sin^2\frac{n\pi x}{L}dx = \frac{L}{2}$$

$$\bar{\varphi}_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \qquad n = 1, 2, \cdots$$

$\{\bar{\varphi}_n\}$ is an orthonormal set, i.e.,

$$\int_0^L \bar{\varphi}_n^2(x)dx = 1$$

$$\int_0^L \bar{\varphi}_m(x)\bar{\varphi}_n(x)dx = 0 \qquad m \neq n$$

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## Expansion of Arbitrary Functions in Series of Orthogonal Functions

Suppose that the set of functions $\{\varphi_n\}$ is orthogonal in a given interval $[a, b]$ w.r.t. $\tilde{r}(x)$. We want to expand a given function $f(x)$ in terms of $\varphi_n$, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{34}$$

Assume that such an expansion exits, multiply both side by $\tilde{r}(x)\varphi_k(x)$ $(k = 0, 1, 2, \cdots)$

$$\tilde{r}(x)\varphi_k(x)f(x) = \sum_{n=0}^{\infty} a_n \tilde{r}(x)\varphi_k(x)\varphi_n(x) \tag{35}$$

and integrate both side over $[a, b]$, i.e.

$$\int_a^b \tilde{r}(x)\varphi_k(x)f(x)dx = \sum_{n=0}^{\infty} a_n \int_a^b \tilde{r}(x)\varphi_k(x)\varphi_n(x)dx \tag{36}$$

Notice that this equation is valid only if equation (34) is *uniformly convergent* in $[a, b]$. Since $\{\varphi_n\}$ is a set of orthogonal functions

$$a_k = \frac{\int_a^b \tilde{r}(x)\varphi_k(x)f(x)dx}{\int_a^b \tilde{r}(x)\varphi_k^2(x)dx} \tag{37}$$

## Proper Sturm-Liouville Problem

A *proper* Sturm-Liouville problem is defined by equation (15) if

- $\tilde{p}(x) > 0$, $\tilde{q}(x) \leq 0$ and $\tilde{r}(x) > 0$ in $[a, b]$;
- Sturm-Liouville condition are satisfied;
- If the boundary condition $y + \alpha \frac{dy}{dx} = 0$ $(\alpha \neq 0)$ is imposed on $x = a$, or $b$, or both, then (1) $\alpha_1 < 0$ at $x = a$, and (2) $\alpha_2 > 0$ at $x = b$.

## Properties of a proper Sturm-Liouville problem

1. For a proper Sturm-Liouville problem,
   - all eigenvalues are real and non-negative and
   - all eigenfunctions are real.
2. If a Sturm-Liouville problem is proper, and if $\tilde{p}(x)$, $\tilde{q}(x)$ and $\tilde{r}(x)$ are *regular* in $(a, b)$, then the representation of a bounded, piecewise differentiable function $f(x)$ in a series of eigenfunctions
   - converges to $f(x)$ inside $[a, b]$ at all points where $f(x)$ is continuous, and
   - converges to the mean value $\frac{1}{2}[f(x+) + f(x-)]$ at points where finite jumps occur.
3. The series may or may not converge to the value of $f(x)$ at end points of the interval, i.e. when $x = a$ or $x = b$.

**[Example]** $f(x) = x = \sum_{n=0}^{\infty} a_n \sin \frac{n\pi}{L} x$

Notice that

$$\frac{d^2y}{dx^2} + \lambda y = 0 \qquad y(0) = y(L) = 0$$

is a proper Sturm-Liouville problem with $\tilde{p}(x) = 1$, $\tilde{q}(x) = 0$ and $\tilde{r}(x) = 1$. The eigenvalues and eigenfunctions previously obtained are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \qquad \varphi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

Thus,

$$a_0 = 0$$

$$a_n \frac{L}{2} = a_n \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx = -L^2 \frac{1}{n\pi}(-1)^n$$

$$a_n = \frac{2L}{\pi} \frac{(-1)^{n+1}}{n} \qquad n = 1, 2, 3, \cdots$$

$$f(x) = x = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

- $x = 0$ :

$$f(x) = 0 \qquad RHS = 0$$

- $x = L$ :

$$f(L) = L \qquad RHS = 0 \neq L$$

- $x = \frac{L}{2}$ :

$$f(L/2) = \frac{L}{2} \qquad RHS = \frac{2L}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\right) = \left(\frac{2L}{\pi}\right)\left(\frac{\pi}{4}\right) = \frac{L}{2}$$

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## BVP Involving Nonhomogeneous Differential Equations

Consider the differential equation

$$\left[\frac{d}{dx}\left(\tilde{p}\frac{dy}{dx}\right) + \tilde{q}y\right] + \lambda\tilde{r}y = F(x) \tag{38}$$

with homogeneous Sturm-Liouville boundary conditions. Here, $\lambda$ is a given constant. This equation can be written in operator notation, i.e.

$$\mathcal{L}y + \lambda\tilde{r}y = F(x) \tag{39}$$

Let us first consider the homogeneous equation

$$\mathcal{L}y + \lambda\tilde{r}y = 0 \tag{40}$$

together with the given BCs. Notice that the corresponding Sturm-Liouville problem results in a set of orthogonal characteristic functions $\{\varphi_n(x)\}$ such that

$$\mathcal{L}\varphi_n(x) + \lambda_n\tilde{r}(x)\varphi_n(x) = 0 \tag{41}$$

Now let us assume that the solution of equation (38) exists. This solution $y(x)$ can be expanded in the form

$$y(x) = \sum_{n=0}^{\infty} a_n\varphi_n(x) \tag{42}$$

Substituting this expression into equation (39) yields

$$\mathcal{L}\left[\sum_n (a_n\varphi_n)\right] + \lambda\tilde{r}\sum_n (a_n\varphi_n) = F(x) \tag{43}$$

From equation (41),

$$\mathcal{L}\left[\sum_n (a_n\varphi_n)\right] + \tilde{r}\sum_n (\lambda_n a_n\varphi_n) = 0 \tag{44}$$

Subtract equation (44) from equation (43), i.e.

$$\tilde{r}(x)\sum_{n=0}^{\infty}(\lambda - \lambda_n)a_n\varphi_n(x) = F(x) \tag{45}$$

Let

$$f(x) = \frac{F(x)}{\tilde{r}(x)} = \sum_{n=0}^{\infty} A_n\varphi_n(x) \qquad \text{and} \qquad A_n = a_n(\lambda - \lambda_n) \tag{46}$$

If $f(x)$ is piecewise differentiable, the $A_n$ can be determined. Thus,

$$y(x) = \frac{A_0}{\lambda - \lambda_0}\varphi_0(x) + \frac{A_1}{\lambda - \lambda_1}\varphi_1(x) + \cdots \tag{47}$$

From the above results, one can draw two important conclusions:

1. If $F(x) \equiv 0$ in $[a, b]$, then from equation (45) $\lambda = \lambda_k$ and $k = 0, 1, 2, \cdots$.
2. If $F(x)$ is not identically zero in $[a, b]$, equation (47) shows that equation (38) has a solution only when $\lambda \neq \lambda_k$ $(k = 0, 1, 2, \cdots)$.

## Fourier Series

Since the BVP

$$\frac{d^2y}{dx^2} + \lambda y = 0 \tag{48}$$
$$y(0) = y(L) = 0 \tag{49}$$

has the eigenfunctions

$$\varphi_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \cdots \tag{50}$$

From the previous discussions, a function can be expressed as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \tag{51}$$

which is referred to as the *Fourier sine series representation* of $f(x)$ in $(0, L)$. The coefficients $a_n$ can be determined by

$$a_n = \frac{\int_0^L f(x)\sin\frac{n\pi x}{L}dx}{\int_0^L \sin^2\frac{n\pi x}{L}dx} \tag{52}$$

Notice that

$$\int_0^L \sin^2\frac{n\pi x}{L}dx = \frac{1}{2}\int_0^L \left(1 - \cos 2\frac{n\pi x}{L}\right)dx = \frac{L}{2} \tag{53}$$

Thus,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{54}$$

It can be observed from equation (51) that all terms of the RHS

1. are *periodic* and have the common period of $2L$;
2. are odd functions, i.e.,

   $$F(-x) = -F(x) \tag{55}$$

   Notice that

   $$\sin\left(-\frac{n\pi x}{L}\right) = -\sin\left(\frac{n\pi x}{L}\right)$$

It follows that in the interval $(-L, 0)$ the series in equation (51) represents the function $-f(-x)$, i.e.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \qquad x \in (0, L) \tag{56}$$

Let $x' = -x$ and $x' \in (-L, 0)$. Then

$$f(-x') = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi(-x')}{L} = -\sum_{n=1}^{\infty} a_n \sin \frac{n\pi x'}{L} \tag{57}$$

As a result,

$$-f(-x') = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x'}{L} \qquad x' \in (-L, 0) \tag{58}$$

If $f(x)$ is an odd function, i.e., $f(-x) = -f(x)$, then

$$f(x') = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x'}{L} \qquad x' \in (-L, 0) \tag{59}$$

Therefore,

- Equation (51) represents $f(x)$ in $(-L, L)$ if $f(x)$ is an odd function.
- If $f(x)$ is also periodic of period $2L$, then equation (51) represents $f(x)$ everywhere.

**[Example] Express $f(x) = e^x$ in $(0, \pi)$ with $\varphi_n(x) = \sin nx$**

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^x \sin(nx)dx = \frac{2}{\pi}\frac{n}{n^2+1}\left(1 - e^{\pi}\cos n\pi\right)$$

$$a_n = \frac{2}{\pi}\frac{n}{n^2+1}\left[1 + e^{\pi}(-1)^{n+1}\right]$$

Thus,

$$\frac{2}{\pi}\left[\frac{e^{\pi}+1}{2}\sin x - \frac{2(e^{\pi}-1)}{5}\sin 2x + \frac{3(e^{\pi}+1)}{10}\sin 3x - \cdots\right] = \begin{cases} e^x & x \in (0, \pi) \\ -e^{-x} & x \in (-\pi, 0) \end{cases}$$

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**[Exercise] Express the odd functions $x$ and $x^3$ in $(-L, L)$ with $\varphi_n(x) = \sin \frac{n\pi x}{L}$**

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Similar series involving cosine functions can be obtained by considering the following BVP:

$$\frac{d^2y}{dx^2} + \lambda y = 0 \tag{60}$$

$$y'(0) = y'(L) = 0 \tag{61}$$

The corresponding eigenfunctions are:

$$\varphi_n(x) = \cos\frac{n\pi x}{L} \tag{62}$$

where, $n = 0, 1, 2, \cdots$. Notice that $\varphi_0(x) = 1$ is a member of the orthogonal set. Thus,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad x \in (0, L) \tag{63}$$

where,

$$a_0 = \frac{\int_0^L f(x)dx}{\int_0^L dx} = \frac{1}{L}\int_0^L f(x)dx \tag{64}$$

$$a_n = \frac{\int_0^L f(x)\cos\frac{n\pi x}{L}dx}{\int_0^L \cos^2\frac{n\pi x}{L}dx} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \tag{65}$$

Equation (63) is known as the *Fourier cosine series* representation of $f(x)$ in $(0, L)$. Since the RHS of equation (63) is an even function, i.e., $\cos(-\frac{n\pi x}{L}) = \cos(\frac{n\pi x}{L})$, then

$$f(-x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad x \in (-L, 0) \tag{66}$$

If $f(x)$ is an even function, i.e., $f(x) = f(-x)$, then $f(x)$ can be represented by equation (63) in $(-L, L)$, i.e.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad x \in (-L, L) \tag{67}$$

**[Example] Express $f(x) = e^x$ in $(0, \pi)$ with $\varphi_n(x) = \cos nx$**

$$a_0 = \frac{1}{\pi}\int_0^\pi e^x dx = \frac{1}{\pi}(e^\pi - 1)$$

$$a_n = \frac{2}{\pi}\int_0^\pi e^x \cos nx dx = \frac{2}{\pi}\frac{1}{n^2+1}(e^\pi \cos n\pi - 1) = \frac{2}{\pi}\frac{1}{n^2+1}[e^\pi(-1)^n - 1]$$

$$\frac{2}{\pi}\left(\frac{e^\pi - 1}{2} - \frac{e^\pi + 1}{2}\cos x + \frac{e^\pi - 1}{5}\cos 2x - \frac{e^\pi + 1}{10}\cos 3x + \cdots\right) = \begin{cases} e^x & x \in (0, \pi) \\ e^{-x} & x \in (-\pi, 0) \end{cases}$$

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Any given function $f(x)$ can be expressed as the sum of an even and an odd function, i.e.

$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right] = f_{even}(x) + f_{odd}(x) \tag{68}$$

One can express these two functions separately as

$$f_{even}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad x \in (-L, L) \tag{69}$$

$$f_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad x \in (-L, L) \tag{70}$$

where,

$$a_0 = \frac{1}{L}\int_0^L f_{even}(x)dx = \frac{1}{2L}\int_{-L}^L f_{even}(x)dx = \frac{1}{2L}\left[\int_{-L}^L f(x)dx - \int_{-L}^L f_{odd}(x)dx\right] \tag{71}$$

$$\Rrightarrow \qquad a_0 = \frac{1}{2L}\int_{-L}^L f(x)dx \tag{72}$$

$$a_n = \frac{2}{L}\int_0^L f_{even}(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^L f_{even}(x)\cos\frac{n\pi x}{L}dx \tag{73}$$

$$\Rrightarrow \qquad a_n = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}dx \tag{74}$$

$$b_n = \frac{2}{L}\int_0^L f_{odd}(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^L f_{odd}(x)\sin\frac{n\pi x}{L}dx \tag{75}$$

$$\Rrightarrow \qquad b_n = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}dx \tag{76}$$

Thus,

$$f(x) = f_{even}(x) + f_{odd}(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \tag{77}$$

Equation (77) is the complete Fourier series representation of $f(x)$ in the interval $(-L, L)$. If $f(x)$ is an even function, $b_n = 0$. If $f(x)$ is an odd function, $a_0 = a_n = 0$. If $f(x)$ is neither even nor odd, then none of the coefficients are zero.

**[Example] Express $f(x) = e^x$ in $(-\pi, +\pi)$ with complete Fourier series representation.**

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^x dx = \frac{1}{2\pi}(e^{\pi} - e^{-\pi}) = \frac{1}{\pi}\sinh\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} e^x \cos nx dx = \frac{2}{\pi}\frac{\cos n\pi}{n^2+1}\sinh\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} e^x \sin nx dx = -\frac{2}{\pi}\frac{n\cos n\pi}{n^2+1}\sinh\pi$$

$$e^x = \frac{\sinh\pi}{\pi}\left[1 - 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+1}(\cos nx - n\sin nx)\right]$$

## Bessel Series

Consider the modified Bessel equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\mu^2 x^2 - p^2)y = 0 \tag{78}$$

in the interval $(0, L)$. This equation can be written as

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(-\frac{p^2}{x} + \mu^2 x\right) y = 0 \tag{79}$$

If this equation is compared with

$$\frac{d}{dx}\left[\tilde{p}(x)\frac{dy}{dx}\right] + \left[\tilde{q}(x) + \lambda\tilde{r}(x)\right] y = 0$$

then it can be observed that

$$\tilde{p}(x) = x$$

$$\tilde{q}(x) = -\frac{p^2}{x}$$

$$\tilde{r}(x) = x$$

$$\lambda = \mu^2$$

Compare with

$$\frac{d}{dx}\left[\tilde{p}(x)\frac{dy}{dx}\right] + \left[\tilde{q}(x) + \lambda\tilde{r}(x)\right] y = 0 \tag{80}$$

We can conclude that

$$\tilde{p}(x) = x \qquad \tilde{q}(x) = -p^2/x \qquad \tilde{r}(x) = x \qquad \lambda = \mu^2 \tag{81}$$

The general solution of equation (78) is of the following form

$$y(x) = \begin{cases} c_1 J_p(\mu x) + c_2 J_{-p}(\mu x) & \text{if } p \text{ is not an integer} \\ c_1 J_p(\mu x) + c_2 Y_p(\mu x) & \text{if } p \text{ is a non-negative interger} \end{cases} \tag{82}$$

Let us consider the interval $(0, L)$. It is clear that $\tilde{p}(0) = 0$. Thus, the eigenfunctions of the problem are orthogonal in $(0, L)$ w.r.t. $\tilde{r}(x) = x$, if

1. $x = 0$

$$y(0) = \text{finite} \tag{83}$$
$$y'(0) = \text{finite} \tag{84}$$

2. $x = L$

   One of the Sturm-Liouville conditions must be satisfied, i.e.,

$$y(L) = 0 \tag{85}$$

or

$$y'(L) = 0 \tag{86}$$

or

$$y'(L) + ky(L) = 0 \qquad k \geq 0 \tag{87}$$

Since $y(0) =$ finite, then $c_2 = 0$ due to the fact that $J_{-p}(0)$ and $Y_p(0)$ are not finite. Thus,

$$y(x) = c_1 J_p(\mu x) \tag{88}$$

- If $y(L) = 0$, then

$$J_p(\mu_n L) = 0 \tag{89}$$

- If $y'(L) = 0$, then

$$J'_p(\mu_n L) = 0 \tag{90}$$

- If $y'(L) + ky(L) = 0$, then

$$J'_p(\mu_n L) + kJ_p(\mu_n L) = 0 \tag{91}$$

In all three cases, the eigenfunctions are of the form

$$\varphi_n(x) = J_p(\mu_n x) \tag{92}$$

where $\mu_n$ is the solution of one of the equations (89), (90) and (91). As a result, these functions are orthogonal in $(0, L)$ w.r.t. $\tilde{r}(x) = x$, i.e.,

$$\int_0^L x J_p(\mu_m x) J_p(\mu_n x) dx = 0 \tag{93}$$

where $m \neq n$.

Not all the eigenfunctions corresponding to a given $p$ are needed in the orthogonal set. This is due to the facts that

1. Since $J_p(-x) = (-1)^p J_p(x)$, the solution of equations (89), (90) and (91) exist in pairs, symmetrically located w.r.t. $x = 0$. On the other hand,

$$\varphi_n(x) = J_p(\mu_n x) = (-1)^p J_p(-\mu_n x) = (-1)^p J_p(\mu_m x) = (-1)^p \varphi_m(x) \tag{94}$$

   where $\mu_m = -\mu_n$. Thus, $\varphi_n(x)$ an $\varphi_m(x)$ are linearly dependent and negative value of $\mu_n$ need not be considered.

2. If $\mu_0 = 0$, there are two possible cases to be considered:

   (a) $p > 0$
   Notice that

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+p}}{k!\Gamma(k+p+1)} \tag{95}$$

$$\varphi_0(x) = J_p(\mu_0 x) = J_p(0) = 0 \tag{96}$$

   Thus, $\varphi_0(x)$ can not be an eigenfunction.

   (b) $p = 0$
   Note that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots \tag{97}$$

   Thus, only equation (90) is possible and $\varphi_0(x) = J_0(\mu_0 x) = J_0(0) = 1$.

**Conclusion:** It is necessary to consider only the set of functions $\{\varphi_n(x)\}$ corresponding to *positive* values of $\mu_n$ $(n = 1, 2, 3, \cdots)$ in all cases, except in the case of equation (90) when $p = 0$, in which case the eigenfunction $\varphi_0(x) = 1$ corresponding to $\mu_0 = 0$ must be added to the set.

Let us temporarily exclude the exceptional case, i.e., equation (90) with $p = 0$, and consider the series representation of a function:

$$f(x) = \sum_{n=1}^{\infty} a_n J_p(\mu_n x) \tag{98}$$

where $p > 0$ and $\mu_n$ is the positive solution of equation (89), (90) or (91). Since the functions in the series form an orthogonal set,

$$a_n = \frac{1}{c_n} \int_0^L x f(x) J_p(\mu_n x) dx \tag{99}$$

where

$$c_n = \int_0^L x \left[J_p(\mu_n x)\right]^2 dx \tag{100}$$

To determine $c_n$, we have to go through an indirect route. First, substitute a characteristic function $\varphi_n(x)$ in equation (79):

$$\frac{d}{dx}\left(x\frac{d\varphi_n}{dx}\right) + \left(-\frac{p^2}{x} + \mu_n^2 x\right)\varphi_n = 0 \tag{101}$$

and then multiply both sides by $2x\varphi_n'$

$$2x\frac{d\varphi_n}{dx}\frac{d}{dx}\left(x\frac{d\varphi_n}{dx}\right) + 2x\frac{d\varphi_n}{dx}\left(-\frac{p^2}{x} + \mu_n^2 x\right)\varphi_n = 0 \tag{102}$$

Thus,

$$\left(\mu_n^2 x^2 - p^2\right)\frac{d}{dx}\left(\varphi_n^2\right) = -\frac{d}{dx}\left[\left(x\frac{d\varphi_n}{dx}\right)^2\right] \tag{103}$$

Integrate both sides over $(0, L)$:

$$LHS = \int_0^L \left(\mu_n^2 x^2 - p^2\right)\frac{d}{dx}\left(\varphi_n^2\right)dx = \left[\left(\mu_n^2 x^2 - p^2\right)\varphi_n^2\right]_0^L - 2\mu_n^2 \int_0^L x\varphi_n^2 dx \tag{104}$$

$$RHS = -\int_0^L \frac{d}{dx}\left[\left(x\frac{d\varphi_n}{dx}\right)^2\right]dx = -\left[x^2\left(\frac{d\varphi_n}{dx}\right)^2\right]_0^L = -\left[x^2\left(\frac{d\varphi_n}{dx}\right)^2\right]_{x=L} \tag{105}$$

In equation (104), notice that $\varphi_n(x) = J_p(\mu_n x)$ and

$$\left[\left(\mu_n^2 x^2 - p^2\right)\varphi_n^2\right]_{x=0} = 0 \tag{106}$$

This is because

1. If $p > 0$, $\varphi_n(0) = J_p(0) = 0$.
2. If $p = 0$, $\varphi_0(0) = J_0(0) = 1$. But $\mu_n^2 x^2 - p^2 = 0 - 0 = 0$.

Thus,

$$c_n = \int_0^L x \left[J_p(\mu_n x)\right]^2 dx = \frac{1}{2\mu_n^2} \left\{ \left(\mu_n^2 x^2 - p^2\right) \left[J_p(\mu_n x)\right]^2 + x^2 \left[\frac{d}{dx} J_p(\mu_n x)\right]^2 \right\}_{x=L} \tag{107}$$

The derivative of $J_p(\mu_n x)$ in the above equation can be obtained with the identity

$$\frac{d}{dx} J_p(\mu_n x) = -\mu_n J_{p+1}(\mu_n x) + \frac{p}{x} J_p(\mu_n x) \tag{108}$$

- If equation (89) is satisfied, i.e., $J_p(\mu_n L) = 0$, then from equations (107) and (108) we can get

$$c_n = \frac{L^2}{2} \left[J_{p+1}(\mu_n L)\right]^2 \tag{109}$$

- If equation (90) is satisfied , i.e., $J_p'(\mu_n L) = 0$, the from equation (107) we get

$$c_n = \frac{\mu_n^2 L^2 - p^2}{2\mu_n^2} \left[J_p(\mu_n L)\right]^2 \tag{110}$$

- If equation (91) is satisfied , i.e., $J_p'(\mu_n L) = -k J_p(\mu_n L)$, the from equation (107) we get

$$c_n = \frac{(\mu_n^2 + k^2) L^2 - p^2}{2\mu_n^2} \left[J_p(\mu_n L)\right]^2 \tag{111}$$

Now let's turn to the exceptional case, i.e., equation (90) with $p = 0$. In other words,

$$J_0'(\mu_n L) = 0 \tag{112}$$

Specifically, all solutions of equation (112) have to be considered, including $\mu_0 = 0$. The series representation becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n J_0(\mu_n x) \tag{113}$$

where

$$a_0 = \frac{\int_0^L x f(x) dx}{\int_0^L x dx} = \frac{2}{L^2} \int_0^L x f(x) dx \tag{114}$$

while $a_n$ can be determined with the method described previously with $c_n$ given by equation (110).

**[Example] Express $f(x) = 1$ in interval $(0, L)$ by Bessel series of order zero. The eigenvalues are the solutions of $J_0(\mu_n L) = 0$.**

$$f(x) = 1 = \sum_{n=1}^{\infty} a_n J_0(\mu_n x)$$

where $\alpha_n = \mu_n L$ satisfies

$$J_0(\alpha_n) = 0$$

From table

$$\alpha_1 = 2.4048, \qquad \alpha_2 = 5.5201, \qquad \alpha_3 = 8.6537,$$
$$\alpha_4 = 11.7915, \qquad \alpha_5 = 14.9309, \qquad \alpha_6 = 18.0711$$
$$\cdots$$

$$a_n = \frac{1}{c_n} \int_0^L x J_0(\mu_n x) dx$$

From equation (109),

$$c_n = \frac{L^2}{2} \left[ J_1(\mu_n L) \right]^2$$

From the integral property of $J_p$:

$$\int \eta x^p J_{p-1}(\eta x) dx = x^p J_p(\eta x)$$

Let $\eta = \mu_n$ and $p = 1$,

$$\int \mu_n x J_0(\mu_n x) dx = x J_1(\mu_n x)$$

Thus,

$$\int_0^L x J_0(\mu_n x) dx = \frac{x}{\mu_n} J_1(\mu_n x) \Big|_0^L = \frac{L}{\mu_n} J_1(\mu_n L)$$

$$a_n = \frac{(L/\mu_n) J_1(\mu_n L)}{(L^2/2) \left[ J_1(\mu_n L) \right]^2} = \frac{2}{\mu_n L} \frac{1}{J_1(\mu_n L)}$$

$$f(x) = 1 = \left( \frac{2}{L} \right) \sum_{n=1}^{\infty} \frac{J_0(\mu_n x)}{\mu_n J_1(\mu_n L)}$$

■

**[Example] Express $f(x) = 1-x^2$ in interval $(0,1)$ by Bessel series of order zero. The eigenvalues are the solutions of $J_0(\mu_n) = 0$.**

$$f(x) = 1 - x^2 = \sum_{n=1}^{\infty} a_n J_0(\mu_n x)$$

where $\alpha_n = \mu_n$ satisfies

$$J_0(\alpha_n) = 0$$

From table

$$\alpha_1 = 2.4048, \qquad \alpha_2 = 5.5201, \qquad \alpha_3 = 8.6537,$$

$$\alpha_4 = 11.7915, \qquad \alpha_5 = 14.9309, \qquad \alpha_6 = 18.0711$$

$$\cdots$$

$$a_n = \frac{1}{c_n} \int_0^1 x(1-x^2) J_0(\mu_n x) dx$$

From equation (109),

$$c_n = \frac{1}{2} \left[ J_1(\mu_n) \right]^2$$

From the integral property of $J_p$:

$$\int \eta x^p J_{p-1}(\eta x) dx = x^p J_p(\eta x)$$

Let $\eta = \mu_n$ and $p = 1$,

$$\int \mu_n x J_0(\mu_n x) dx = x J_1(\mu_n x)$$

Let $\eta = \mu_n$ and $p = 2$,

$$\int \mu_n x^2 J_1(\mu_n x) dx = x^2 J_2(\mu_n x)$$

$$\int_0^1 x(1-x^2) J_0(\mu_n x) dx = \int_0^1 x J_0(\mu_n x) dx - \int_0^1 x^3 J_0(\mu_n x) dx$$

$$\int_0^1 x J_0(\mu_n x) dx = \frac{x}{\mu_n} J_1(\mu_n x) \bigg|_0^1 = \frac{1}{\mu_n} J_1(\mu_n)$$

$$\int_0^1 x^3 J_0(\mu_n x) dx = x^2 \frac{x}{\mu_n} J_1(\mu_n x) \bigg|_0^1 - \int_0^1 \frac{x}{\mu_n} J_1(\mu_n x)(2x) dx$$

$$= \frac{1}{\mu_n} J_1(\mu_n) - \frac{2}{\mu_n} \int_0^1 x^2 J_1(\mu_n x) dx$$

$$= \frac{1}{\mu_n} J_1(\mu_n) - \frac{2}{\mu_n^2} J_2(\mu_n)$$

Thus,

$$\int_0^1 x(1-x^2)J_0(\mu_n x)dx = \frac{2}{\mu_n^2}J_2(\mu_n)$$

$$a_n = \frac{\frac{2}{\mu_n^2}J_2(\mu_n)}{\frac{1}{2}\left[J_1(\mu_n)\right]^2} = \frac{4J_2(\mu_n)}{\mu_n^2 J_1^2(\mu_n)}$$

**[Example] Express** $f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$ **in interval** $(0,2)$ **by Bessel series of order 1. The eigenvalues are the solutions of** $J_1(2\mu_n) = 0$**.**

Let $\alpha_n = 2\mu_n$. From table,

$$\alpha_1 = 3.8317 \qquad \alpha_2 = 7.0156 \qquad \alpha_3 = 10.1735$$
$$\alpha_4 = 13.3237 \qquad \alpha_5 = 16.4706$$

The series we seek is

$$f(x) = \sum_{n=1}^{\infty} a_n J_1(\mu_n x), \qquad 0 < x < 2$$

where

$$a_n = \frac{1}{c_n}\int_0^2 x f(x) J_1(\mu_n x) dx$$

and from equation (109)

$$c_n = \frac{L^2}{2}\left[J_{p+1}(\mu_n L)\right]^2 = 2J_2^2(2\mu_n)$$

From the integral property of $J_p$:

$$\int \eta x^p J_{p-1}(\eta x)dx = x^p J_p(\eta x)$$

Let $\eta = \mu_n$ and $p = 2$,

$$\int \mu_n x^2 J_1(\mu_n x)dx = x^2 J_2(\mu_n x)$$

$$\int_0^2 x f(x) J_1(\mu_n x)dx = \int_0^1 x^2 J_1(\mu_n x)dx = \frac{x^2}{\mu_n}J_2(\mu_n x)\Bigg|_0^1 = \frac{1}{\mu_n}J_2(\mu_n)$$

Thus,

$$a_n = \frac{\frac{1}{\mu_n}J_2(\mu_n)}{2J_2^2(2\mu_n)} = \frac{J_2(\mu_n)}{2\mu_n J_2^2(2\mu_n)}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{J_2(\mu_n)}{2\mu_n J_2^2(2\mu_n)} J_1(\mu_n x), \qquad 0 < x < 2$$

## Legendre Series

Consider the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0 \tag{115}$$

where $x \in (-1,1)$. This equation can be written as

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + p(p+1)y = 0 \tag{116}$$

Let us compare this equation with

$$\frac{d}{dx}\left[\tilde{p}(x)\frac{dy}{dx}\right] + \left[\tilde{q}(x) + \lambda\tilde{r}(x)\right]y = 0 \tag{117}$$

Thus, $\tilde{p}(x) = 1-x^2$, $\tilde{q}(x) = 0$, $\tilde{r}(x) = 1$ and $\lambda = p(p+1)$.

From the fact that $\tilde{p}(\pm 1) = 0$, we can conclude that, if $y(\pm 1) =$ finite and $y'(\pm 1) =$ finite, then any two distinct roots of Legendre equation are orthogonal w.r.t $\tilde{r}(x) = 1$ in the interval $(-1,+1)$. Since the solutions of Legendre equation are finite at $x = \pm 1$ only if $p$ is a positive integer or zero, it is only necessary to consider $p = 0, 1, 2, \cdots$. Thus, the eigenfunctions are

$$\varphi_n(x) = P_n(x) = \text{ Legendre Polynomial} \tag{118}$$

The corresponding orthogonality condition is

$$\int_{-1}^{+1} P_m(x)P_n(x)dx = 0 \qquad (m \neq n) \tag{119}$$

A function $f(x)$ which is piecewise differentiable in the interval $(-1,+1)$ can be represented by

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{120}$$

where

$$a_n = \frac{\int_{-1}^{+1} f(x)P_n(x)dx}{\int_{-1}^{+1} P_n^2(x)dx} \tag{121}$$

Let us substitute the Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{122}$$

into the integral

$$\int_{-1}^{+1} g(x)P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{+1} g(x)\frac{d^n}{dx^n}(x^2-1)^n dx \tag{123}$$

Assuming the first $k$ derivatives for $g(x)$ exist and continuous in $(-1, +1)$ and noticing that

$$\frac{d}{dx}(x^2-1)^n = \frac{d^2}{dx^2}(x^2-1)^n = \cdots = \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = 0 \tag{124}$$

at $x = \pm 1$, one can integrate equation (123) by parts $k$ times ($k \leq n$) to obtain

$$\int_{-1}^{+1} g(x)P_n(x)dx = \frac{(-1)^k}{2^n n!}\int_{-1}^{+1}\left[\frac{d^k}{dx^k}g(x)\right]\left[\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n\right]dx \tag{125}$$

When $k = n$,

$$\int_{-1}^{+1} g(x)P_n(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{+1}(x^2-1)^n\frac{d^n g(x)}{dx^n}dx \tag{126}$$

Let

$$g(x) = P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1) \tag{127}$$

Thus,

$$\frac{d^n g(x)}{dx^n} = \frac{d^n P_n(x)}{dx^n} = \frac{1}{2^n n!}\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = \frac{(2n)!}{2^n n!} \tag{128}$$

It can also be shown that

$$\int_{-1}^{+1}(1-x^2)^n dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!} \tag{129}$$

Substituting equations (128) and (129) into equation (126) yields

$$\int_{-1}^{+1} P_n^2(x)dx = \frac{2}{2n+1} \tag{130}$$

Consequently, equation (121) can be written as

$$a_n = \frac{2n+1}{2}\int_{-1}^{+1} f(x)P_n(x)dx = \frac{2n+1}{2^{n+1}n!}\int_{-1}^{+1}(1-x^2)^n\frac{d^n f(x)}{dx^n}dx \tag{131}$$

Notice that $P_n(x)$ is an even function if $n$ is even and $P_n(x)$ is an odd function if $n$ is odd. Thus, if $f(x)$ is an even function

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (2n+1)\int_0^{+1} f(x)P_n(x)dx & \text{if } n \text{ is even} \end{cases} \tag{132}$$

On the other hand, if $f(x)$ is an odd function, then

$$a_n = \begin{cases} (2n+1)\int_0^{+1} f(x)P_n(x)dx & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{133}$$

From equation (131), one can also see that any polynomial of degree $N$ can be expressed as a linear combination of the first $N+1$ Legendre polynomials.

**[Example] Express $f(x) = x^2$ as a Legendre series in $(-1, +1)$.**

Since $x^2$ is a polynomial,

$$x^2 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$$
$$a_3 = a_4 = a_5 = \cdots = 0$$

This is due to

$$\frac{d^k f(x)}{dx^k} = 0$$

for $k = 3, 4, \cdots$.

Since $f(x) = x^2$ is even, $a_1 = 0$. Let us make use of equation (131) for $n = 0, 2$, i.e.,

$$a_0 = \frac{1}{2}\int_{-1}^{+1} x^2 dx = \frac{1}{3}$$

$$a_2 = \frac{5}{2^3 2!}\int_{-1}^{+1} (1 - x^2)2dx = \frac{2}{3}$$

Thus,

$$x^2 = \frac{1}{3}\left[P_0(x) + 2P_2(x)\right]$$

where $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$.